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Asymptotic Results on Occupancy Times of Random Walks

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In this note we derive, using Wald's theorem asymptotic results on mean occupancy time of an interval for random walks with arbitrary transition probabilities. We show that our results are consistent with those obtained (by Weiss, Ref. 2) via the master equation approach, by demonstrating that the resulting infinite series can be summed exactly.

KEY WORDS: Lattice random walks; occupation times; first passage times; continuum limit; Wald's identity.

1. INTRODUCTION

Gutkowicz-Krusin *et al.*[1] have derived asymptotic results for the mean occupancy time of a random walk on a one-dimensional lattice with absorbing boundaries. They considered random walks generated by asymmetric exponential unit jump probability density functions. They could obtain exact results because of the particular form of the transition probability they had chosen. In a subsequent paper Weiss,[2] using the master equation approach, has shown that asymptotic results can be obtained for random walks generated by any general unit jump probability density, if its variance is finite.

In this paper we show that exact results can be obtained for a general random walk within the framework of the random walk theory. This we accomplish using Wald's theorem[3] developed for problems that arise in

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sequential analysis. That such a possibility exists has been noticed by Weiss.[2]

It can be seen from this paper that the use of Wald's theorem leads to a simple and straightforward derivation of the mean occupancy time. We found that our expression did not tally with that of Weiss[2] except for the first term. Weiss obtains his expression by taking only the term with $l = 0$, of his infinite series solution (l extending from $-\infty$ to $+\infty$). In this paper we show that the infinite series can be summed exactly leading to an expression for the mean occupancy time which is identical with ours.

2. RANDOM WALK AND THE MOMENT GENERATING FUNCTION

We consider random walk on a line segment with absorbing boundaries at $x = a$ and b, $a < 0 < b$. We work in the continuum limit of the jump size. The random walk is defined by

$$S_n = S_{n-1} + x, \qquad S_0 = 0 \tag{1}$$

where S_n is the position of the random walk after n steps. x is random with a known probability density function, say $f(x)$ (unit jump probability density), that generates the random walk. Let $\mu_1, \mu_2, \mu_3 \ldots$ define the moments of the random walk generator. These can be obtained from the moment generating function denoted by $M(\theta)$, defined as

$$M(\theta) = \int_{-\infty}^{+\infty} \exp(\theta x) f(x)\, dx \tag{1a}$$

We assume $M(\theta)$ to exist for all real values of θ and that $M(\theta) \to \infty$ as $\theta \to \pm\infty$. Then it can be shown[4] that the equation $M(\theta) = 1$ has in general two roots, one denoted by θ_0 the other being zero.

3. MEAN FIRST PASSAGE TIME FOR RANDOM WALKS STARTING FROM ORIGIN

Let us denote by η the random value of n at which the random walk reaches first the left or right boundary. Wald's identity is given by

$$\left\langle \exp(\theta S_\eta)\left[M(\theta)\right]^{-\eta} \right\rangle = 1 \tag{2}$$

We set out to determine $\langle \eta \rangle$, the mean first passage time of the random walk starting from the origin. This is accomplished as follows:

We differentiate with respect to θ, the expression inside the expectation sign of Eq. (2) and set $\theta = 0$. We get

$$\langle \eta \rangle = \frac{\langle S_\eta \rangle}{\mu_1} \tag{3}$$

An expression for $\langle S_\eta \rangle$ is obtained as follows. Let P_a denote the probability that $S_\eta = a$. Hence the probability that $S_\eta = b$ is $1 - P_a$. Here we assume that the random walk reaches the left or right boundaries precisely at $n = \eta$, whereas actually the boundary is crossed in a jump. Nevertheless, this assumption and the expressions derived in this section [Eqs. (4)–(6)] become valid in the asymptotic limit of the individual step becoming small as compared to the distances to be traversed, i.e., when $(b-a) \gg |\mu_1|$ and $(b-a)|\mu_1| \gg \sigma^2$, where $\sigma^2 = \mu_2 - \mu_1^2$. Equation (2) can be written as

$$P_a \left\langle \exp(a\theta)[M(\theta)]^{-\eta} \right\rangle_{S_\eta = a} + (1 - P_a) \left\langle \exp(b\theta)[M(\theta)]^{-\eta} \right\rangle_{S_\eta = b} = 1 \tag{4}$$

Note that in the above the expectations are conditional on S_η. Setting $\theta = \theta_0$ and noting that $M(\theta_0) = 1$ we get

$$P_a = \frac{1 - \exp(b\theta_0)}{\exp(a\theta_0) - \exp(b\theta_0)} \tag{5}$$

using the above, we can write the expression for $\langle S_\eta \rangle$ as

$$\langle S_\eta \rangle = aP_a + b(1 - P_a) = \frac{(a-b)[1 - \exp(b\theta_0)]}{\exp(a\theta_0) - \exp(b\theta_0)} + b \tag{6}$$

4. MEAN OCCUPANCY TIME

We now consider the problem of finding the mean occupancy time T of the random walk with the prescription that its initial position be distributed uniformly throughout the interval (a, b). Let y be the initial point of the random walk. The mean first passage time denoted by $\bar{\eta}(y)$ can be obtained from Eqs. (3) and (6), by recasting them under transformation of origin to y, as

$$\bar{\eta}(y) = \frac{(a-b)\left[1 - \exp[(b-y)\theta_0]\right]}{\mu_1\left\{\exp[(a-y)\theta_0] - \exp[(b-y)\theta_0]\right\}} + \frac{b-y}{\mu_1} \tag{7}$$

Integrating $\bar{\eta}(y)$ from a to b weighted with a uniform density in (a, b) we obtain an expression for T as

$$T = \frac{1}{b-a}\int_a^b \bar{\eta}(y)\,dy = \frac{b-a}{\mu_1} + \frac{1}{\mu_1\theta_0} - \frac{b-a}{\mu_1\left\{1 - \exp[\theta_0(a-b)]\right\}} \tag{8}$$

To get the results in the form given in Ref. 2, we set $a = 0$ and $b = N$. Then

$$T = \frac{N}{2\mu_1} + \frac{1}{\mu_1\theta_0} - \frac{N}{\mu_1[1 - \exp(-N\theta_0)]} \tag{9}$$

As a further approximation, we may treat the random walk generator as being normally distributed. When the individual steps become infinitesimally small, this approximation will be valid. For a Gaussian, we have $\theta_0 = -2\mu_1/\sigma^2$ and we get

$$T = \frac{N}{2\mu_1} - \frac{\sigma^2}{2\mu_1^2} + \frac{N}{\mu_1\left[\exp(2N\mu_1/\sigma^2) - 1\right]} \tag{10}$$

The above expression and those given in Refs. 1 and 2 differ one from the other except for the leading term. The expression for T obtained by Weiss[2] denoted T' is reproduced below for comparison:

$$T' = \frac{N}{2\mu_1} - \frac{\sigma^4}{4N\mu_1^3} + O\left[\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right] \tag{11}$$

To resolve this discrepancy, we redid the calculations of Weiss and these are briefly reported in the Appendix. It is seen that Weiss obtained his result [Eq. (11)] by taking only the term with $l = 0$ of the infinite series solution (l extending from $-\infty$ to $+\infty$) of the partial differential equation, approximating the master equation. We find that the infinite series can be summed exactly and the resulting expression is identical with our results (see Appendix).

5. MEAN OCCUPANCY TIME FOR RANDOM WALKS WITH $\mu_1 = 0$

To derive the results for the case of $\mu_1 = 0$, we differentiate twice with respect to θ the expression inside the expectation integral of Eq. (2) and get

$$\langle \eta \rangle = \langle S_\eta^2 \rangle / \mu_2 \tag{12}$$

An expression for $\langle S_\eta^2 \rangle$ is obtained as

$$\langle S_\eta^2 \rangle = a^2 P_a + b^2(1 - P_a) \tag{13}$$

where P_a is given by

$$P_a = \lim_{\theta_0 \to 0} \frac{1 - \exp(b\theta_0)}{\exp(a\theta_0) - \exp(b\theta_0)} = \frac{b}{b-a} \tag{14}$$

From Eqs. (12), (13), and (14), we get

$$\langle \eta \rangle = -ab/\mu_2 \tag{15}$$

To get an expression for T, we proceed as before by first expressing $\langle \eta \rangle$ as $\bar{\eta}(y)$ in the coordinate system with the origin transferred to y and then integrating $\bar{\eta}(y)$ from a to b weighted by a uniform density of the initial point of random walk. We get

$$T = -\frac{1}{\mu_2(b-a)} \int_a^b (a-y)(b-y)\,dy \tag{16}$$

Considering the case with $a = 0$ and $b = N$, we get

$$T = N^2/6\mu_2 \tag{17}$$

This last result agrees with that reported in Refs. 1 and 2.

6. CONCLUSIONS

A general procedure based on Wald's theorem for determining the mean (and higher moments) of occupancy time of random walks on a one-dimensional lattice with arbitrary jump probability density function is given. The asymptotic results reported here agree with those of Weiss[(2)] obtained via a master equation approach provided we take into account all the terms in his infinite series solution.

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APPENDIX

Weiss[(2)] defines $Q_n(r \mid r_0)$ as the probability that at step n the random walk is at r given that $Q_0(r \mid r_0) = \delta(r - r_0)$. The master equation that $Q_n(r \mid r_0)$ satisfies is approximated to a partial differential equation, which when solved with the boundary conditions that $Q_n(0 \mid r_0) = Q_n(N \mid r_0) = 0$ yields a series solution whose Poisson transformation (see Ref. 2 for details) results in the following expression:

$$\begin{aligned} Q_n(r \mid r_0) &= \frac{1}{(2\pi n\sigma^2)^{1/2}} \exp\left[\frac{2n\mu_1(r - r_0) - \mu_1^2 n^2}{2n\sigma^2}\right] \\ &\quad \times \sum_{l=-\infty}^{+\infty} \left\{ \exp\left[-\frac{(2lN + r - r_0)^2}{2n\sigma^2}\right] - \exp\left[-\frac{(2lN + r + r_0)^2}{2n\sigma^2}\right] \right\} \end{aligned} \tag{A.1}$$

The above expression can be used in the following integration to yield T:

$$T = \frac{1}{N}\int_0^N dr \int_0^N dr_0 \int_0^\infty dn\ Q_n(r \mid r_0) \tag{A.2}$$

Weiss takes the term corresponding to $l = 0$ in Eq. (A.1) and obtains an expression for T as given in Eq. (11).

We show in what follows that the infinite series can be exactly summed.

Starting from the expression for $Q_n(r \mid r_0)$ [see Eq. (A.1)], we calculate T by carrying out the integration [Eq. (A2)] exactly. The intermediate step reads as

$$\begin{aligned} T = {} & \frac{N}{2\mu_1} + \frac{\sigma^2}{2\mu_1^2}\exp\left(\frac{-2N\mu_1}{\sigma^2}\right) - \frac{\sigma^4}{4N\mu_1^3}\left[1-\exp\left(\frac{-2N\mu_1}{\sigma^2}\right)\right] \\ & + \sum_{l=1}^{\infty}\exp\left(-\frac{2l\mu_1 N}{\sigma^2}\right)\left\{\frac{N}{\mu} - \frac{\sigma^2}{2\mu_1^2}\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right]\right\} \\ & + \sum_{l=-1}^{-\infty}\exp\left[\frac{(2l+2)\mu_1 N}{\sigma^2}\right] \\ & \times\left\{\frac{\sigma^4}{4N\mu_1^3}\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right]^2 - \frac{\sigma^2}{2\mu_1^2}\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right]\right\} \end{aligned}$$

After carrying out the summation, we get

$$\begin{aligned} T = {} & \frac{N}{2\mu_1} - \frac{\sigma^4}{4N\mu_1^3}\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right] + \frac{\sigma^2}{2\mu_1^2}\exp\left(-\frac{2N\mu_1}{\sigma^2}\right) \\ & + \left\{\frac{N}{\mu_1} - \frac{\sigma^2}{2\mu_1^2}\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right]\right\}\frac{\exp(-2N\mu_1/\sigma^2)}{1-\exp(-2N\mu_1/\sigma^2)} \\ & + \frac{1}{1-\exp(-2N\mu_1/\sigma^2)} \\ & \times\left\{\frac{\sigma^4}{4N\mu_1^3}\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right]^2 - \frac{\sigma^2}{2\mu_1^2}\times\left[1-\exp\left(-\frac{2N\mu_1}{\sigma^2}\right)\right]\right\} \end{aligned}$$

which reduces to

$$T = \frac{N}{2\mu_1} - \frac{\sigma^2}{2\mu_1^2} + \frac{N}{\mu_1\left[\exp(2N\mu_1/\sigma^2) - 1\right]}$$

The above is exactly the result we have obtained using Wald's theorem [see Eq. (10)].

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